



ENGINEERING MECHANICS I:

AREA MOMENTS OF INERTIA

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Chapter 6

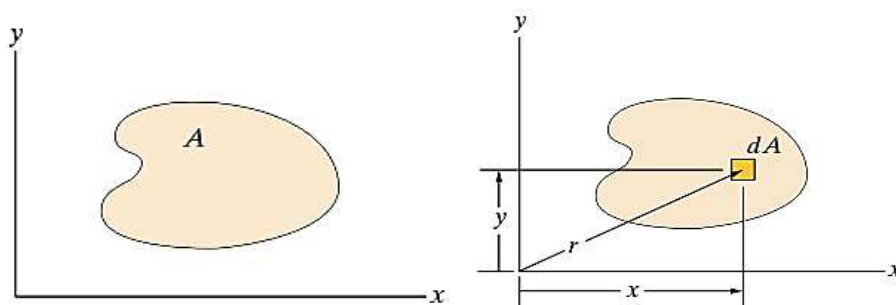
Area Moments of Inertia

6.1. Introduction

The moment of inertia (I) is a geometric property of an area that is used to determine the strength of a structural member or the location of a resultant pressure force acting on a plate submerged in a fluid. It is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as x-x or y-y. It is sometimes referred to as the second moment of the area about an axis, because the distance from the axis to each area element is squared. It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis (axis of interest). The reference axis is usually a centroidal axis.

By definition, the moments of inertia of a differential area dA about the x and y axes are $dlx = y^2 dA$ and $dly = x^2 dA$, respectively. For the entire area A the moments of inertia are determined by integration. The moment of inertia is also known as the Second Moment of the Area.

Consider an area A in the x y plane (Fig.). Four moments of inertia of A are defined as:



a. Moment of inertia about the x axis

$$I_x = \int_A y^2 dA, \quad \text{Where } y \text{ is the } y \text{ coordinate of the differential element of area } dA.$$

b. Moment of inertia about the y axis

$$I_y = \int_A x^2 dA, \quad \text{Where } x \text{ is the } x \text{ coordinate of the element } dA.$$

c. Product of inertia

$$I_{xy} = \int_A xy dA. \quad \text{If an area is symmetric about either the } x \text{ axis or the } y \text{ axis, its product of inertia is zero.}$$

d. Polar moment of inertia

$$J_O = \int_A r^2 dA, \quad \text{Where } r \text{ is the radial distance from the origin of the coordinate system to } dA.$$

The polar moment of inertia is equal to the sum of the moments of inertia about the x and y axes

$$J_O = \int_A r^2 dA = \int_A (y^2 + x^2) dA = I_x + I_y$$

6.2. Radius of Gyration k

The radius is a measure of the distribution of the area from the axis in question. The **radius of gyration** of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics, Provided the areas and moments of inertia are **known**, the radii of gyration are determined from the formulas.

$$k_x = \sqrt{\frac{I_x}{A}}$$

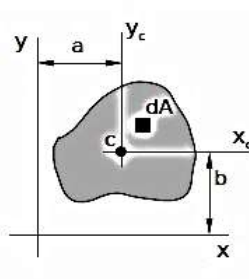
$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

6.3. Parallel Axis Theorem

The parallel-axis theorem can be used to find the moment of inertia of an area about any axis that is parallel to an axis passing through the centroid and about which the moment of inertia is known. In general, If the moment of inertia of an area is known about its centroidal axis, then the moment of inertia about a corresponding parallel axis can be determined using the parallel-axis theorem.

So this theorem provides relationship between centroidal moments of inertia and moments of inertia about parallel axes.



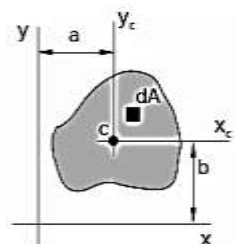
Let $\begin{cases} x = x_c + a \\ y = y_c + b \end{cases}$

Then $\begin{cases} \bar{x}_c = \frac{\int x_c dA}{A} = 0 \\ \bar{y}_c = \frac{\int y_c dA}{A} = 0 \end{cases}$ location of centroid

$$I_x = \int_A y^2 dA = \int_A (y_c + b)^2 dA = \underbrace{\int_A y_c^2 dA}_{I_{xc}} + \underbrace{2b \int_A y_c dA}_0 + \underbrace{b^2 \int_A dA}_{b^2 \cdot A}$$

$$I_x = I_{xc} + b^2 \cdot A$$

Therefore,



$$I_x = I_{xc} + b^2 \cdot A$$

$$I_y = I_{yc} + a^2 \cdot A$$

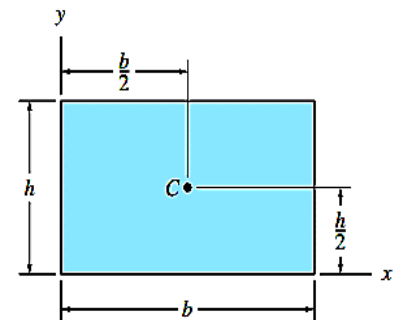
In connection with this, perpendicular axis theorem states that if I_{xx} and I_{yy} be the moment of inertia of a plane section about two mutually perpendicular axis x-x and y-y in the plane of the section, then

the moment of inertia of the section I_{zz} about the axis $z-z$, perpendicular to the plane and passing through the intersection of $x-x$ and $y-y$ is given by:

$I_{zz} = I_{xx} + I_{yy}$ The moment of inertia I_{zz} is also known as polar moment of inertia.

Example 6.1

Determine the area moment of inertia of the rectangle shown in figure on the given axes and its centroidal axes.

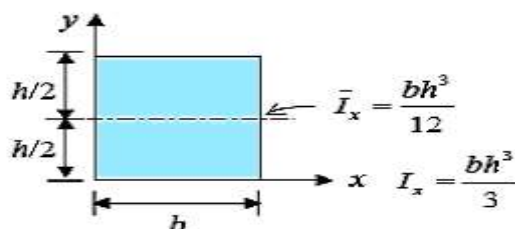


Solution

The moment of inertia of this rectangle can be determined by using direct integration method as follows:

$ \begin{aligned} I_x &= \int_A y^2 dA \\ &= \int_0^h y^2 (b dy) \\ &= \frac{(by^3)}{3} \Big _0^h \\ &= \frac{bh^3}{3} \quad \leftarrow \end{aligned} $	$ \begin{aligned} \bar{I}_x = I_{x'} &= \int_A y'^2 dA \\ &= 4 \int_0^{h/2} y'^2 \left(\frac{b}{2} dy'\right) \\ &= 4 \left(\frac{b}{2}\right) \frac{y'^3}{3} \Big _0^{h/2} \\ &= \frac{bh^3}{12} \quad \leftarrow \end{aligned} $
$ \begin{aligned} I_y &= \int_A x^2 dA \\ &= \int_0^b x^2 (h dx) \\ &= \frac{(hx^3)}{3} \Big _0^b \\ &= \frac{hb^3}{3} \quad \leftarrow \end{aligned} $	$ \begin{aligned} \bar{I}_y = I_{y'} &= \int_A x'^2 dA \\ &= 4 \int_0^{b/2} x'^2 \left(\frac{h}{2} dx'\right) \\ &= 4 \left(\frac{h}{2}\right) \frac{x'^3}{3} \Big _0^{b/2} \\ &= \frac{hb^3}{12} \quad \leftarrow \end{aligned} $

We can also determine the moment of inertia for the given axes x and y by using parallel axis theorem.



$$I_x = \bar{I}_x + Ad^2$$

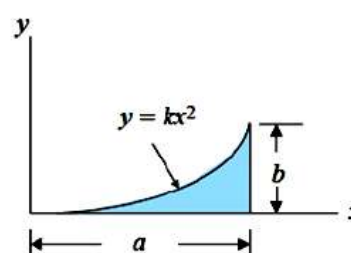
$$= \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2$$

$$= \frac{bh^3}{12} + \frac{bh^3}{4}$$

$$I_x = \frac{bh^3}{3}$$

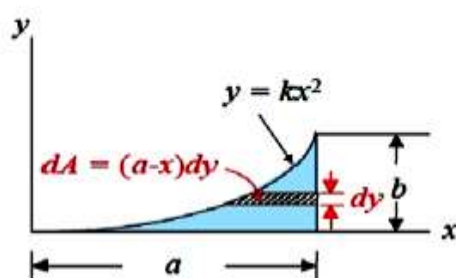
Example 6.2.

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes



Solution

- ✓ Moment of Inertia I_x



Substituting $x = a$ and $y = b$

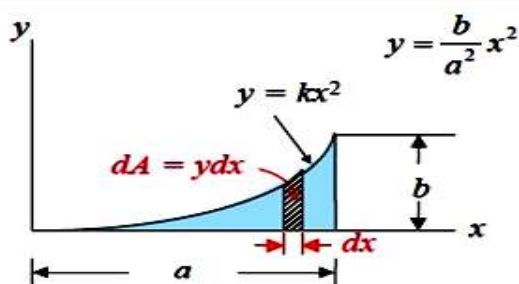
$$y = kx^2$$

$$b = ka^2$$

$$k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad \text{or} \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

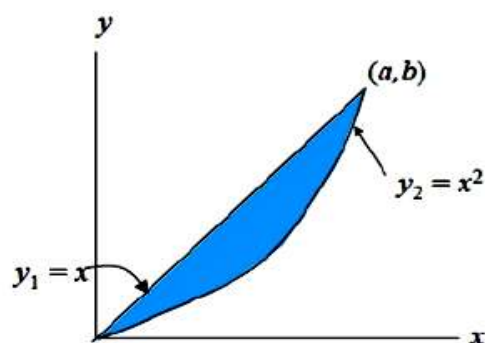
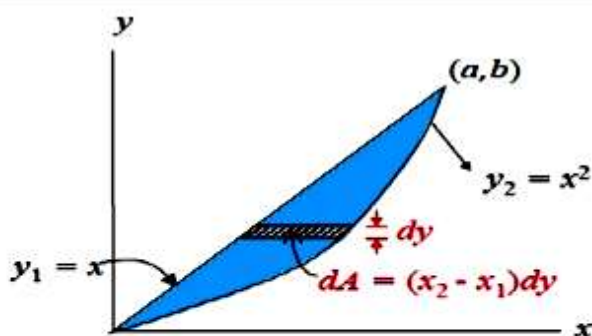
$$\begin{aligned} I_x &= \int_A y^2 dA \\ &= \int_0^b y^2 (a-x) dy \\ &= \int_0^b y^2 \left(a - \frac{a}{b^{1/2}} y^{1/2}\right) dy \\ &= a \int_0^b y^2 dy - \frac{a}{b^{1/2}} \int_0^b y^{5/2} dy \\ &= \frac{ay^3}{3} \Big|_0^b - \frac{a}{b^{1/2}} \left(\frac{2}{7} y^{7/2}\right) \Big|_0^b \\ &= \frac{ab^3}{3} - \frac{a}{b^{1/2}} \left(\frac{2}{7} b^{7/2}\right) \\ &= \frac{ab^3}{3} - \frac{2ab^3}{7} \\ &= \frac{ab^3}{21} \end{aligned}$$

✓ Moment of Inertia I_y 

$$\begin{aligned}
 I_y &= \int_A x^2 dA \\
 &= \int_0^a x^2 y dx \\
 &= \int_0^a x^2 \left(\frac{b}{a^2} x^2 \right) dx \\
 &= \frac{b}{a^2} \int_0^a x^4 dx \\
 &= \left(\frac{b}{a^2} \right) \left(\frac{x^5}{5} \right) \Big|_0^a \\
 &= \left(\frac{b}{a^2} \right) \left(\frac{a^5}{5} \right) \\
 &= \frac{a^3 b}{5} \quad \leftarrow
 \end{aligned}$$

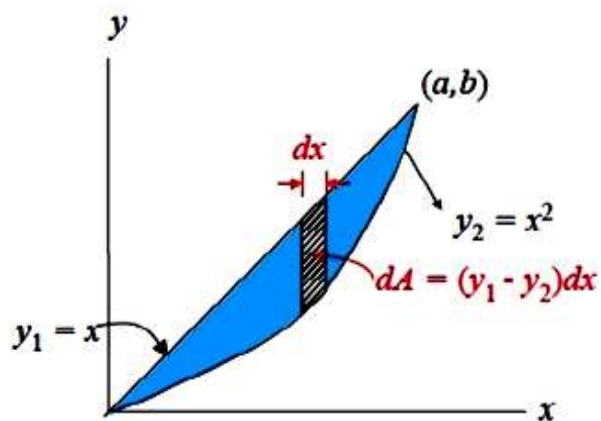
Example 6.3.

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.

**Solution**✓ Moment of Inertia I_x 

$$\begin{aligned}
 I_x &= \int_A y^2 dA \\
 &= \int_0^b y^2 (x_2 - x_1) dy \\
 &= \int_0^b y^2 (y^{1/2} - y) dy \\
 &= \int_0^b (y^{5/2}) dy - \int_0^b (y^3) dy \\
 &= \frac{2}{7} y^{7/2} \Big|_0^b - \frac{y^4}{4} \Big|_0^b \\
 &= \frac{2}{7} b^{7/2} - \frac{b^4}{4} \quad \leftarrow
 \end{aligned}$$

✓ Moment of Inertia I_y

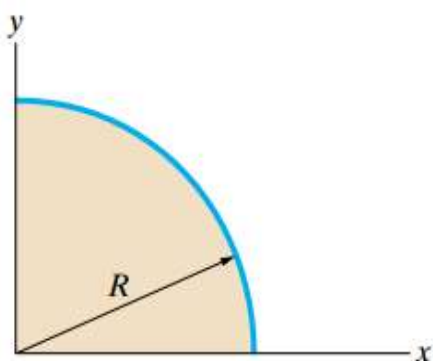


$$\begin{aligned}
 I_y &= \int_A x^2 dA \\
 &= \int_0^a x^2 (y_1 - y_2) dx \\
 &= \int_0^a x^2 (x - x^2) dx \\
 &= \int_0^a (x^3) dx - \int_0^a (x^4) dx \\
 &= \left. \frac{x^4}{4} \right|_0^a - \left. \frac{x^5}{5} \right|_0^a \\
 &= \frac{a^4}{4} - \frac{a^5}{5} \quad \leftarrow
 \end{aligned}$$

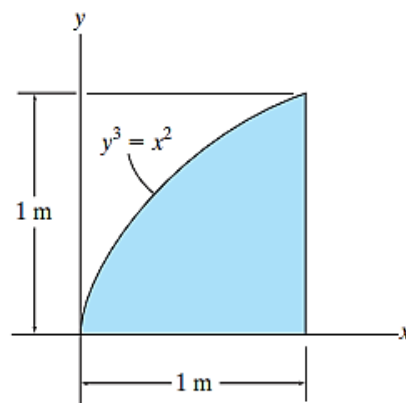
Exercise 6.1.

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes.

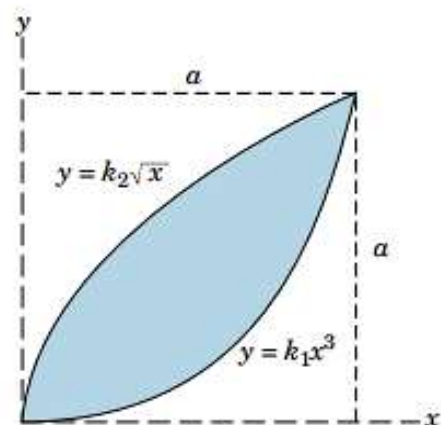
A.



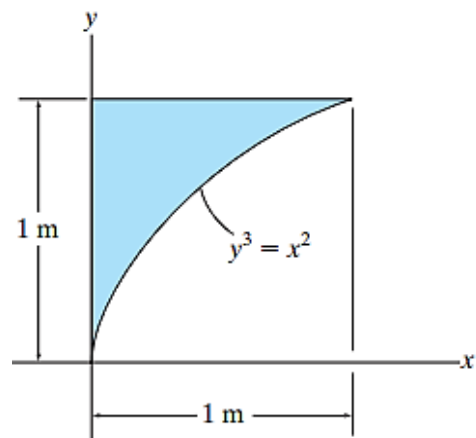
C.



B.



D.



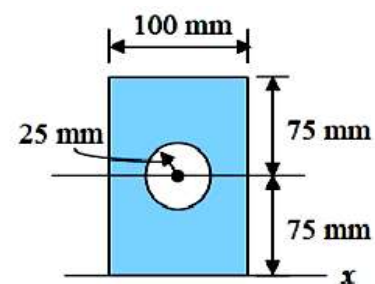
6.4. Moment of Inertia of composite areas

The moment of inertia of a composite area about a particular axis is simply the sum of the moments of inertia of its component parts about the same axis. It is often convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area as a negative quantity.

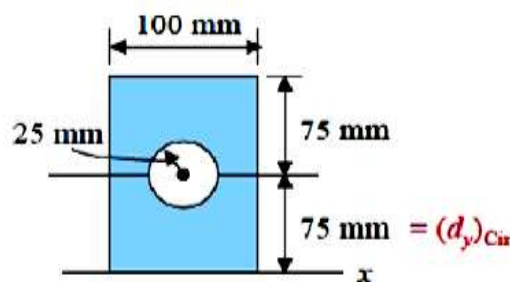
When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A , its centroidal moment of inertia I , the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2 .

Example 6.4.

Compute the moment of inertia of the composite area shown below



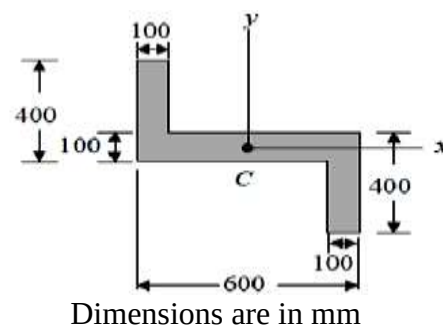
Solution



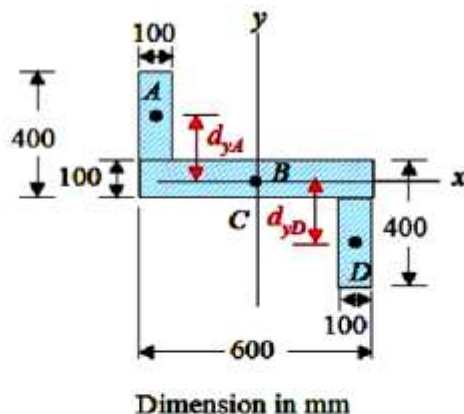
$$\begin{aligned}
 I_x &= \left(\frac{bh^3}{3}\right)_{\text{Rect}} - (\bar{I}_x + Ad_y^2)_{\text{Cir}} \\
 &= \left[\frac{1}{3}(100)(150)^3\right]_{\text{Rect}} - \left[\frac{1}{4}\pi(25)^4 + (\pi \times 25^2)(75)^2\right]_{\text{Cir}} \\
 &= 101 \times 10^6 \text{ mm}^4 \quad \leftarrow
 \end{aligned}$$

Example 6.5.

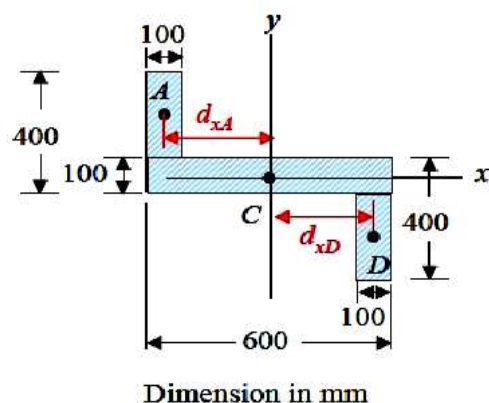
Determine the moments of inertia of the beam's cross-sectional area shown about the x and y centroidal axes



Solution



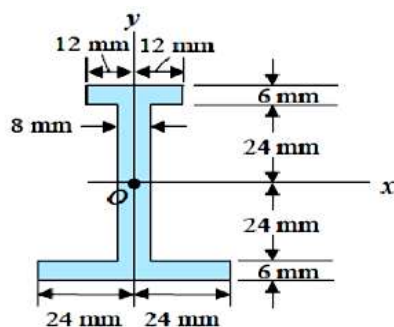
$$\begin{aligned}
 I_x &= (\bar{I}_x + Ad_y^2)_A + (\bar{I}_x + Ad_y^2)_B + (\bar{I}_x + Ad_y^2)_C \\
 &= \left[\frac{1}{12} (100)(300)^3 + (100 \times 300)(200)^2 \right] + \left[\frac{1}{12} (600)(100)^3 + 0 \right] \\
 &\quad + \left[\frac{1}{12} (100)(300)^3 + (100 \times 300)(200)^2 \right] \\
 &= 2.9 \times 10^9 \text{ mm}^4 \quad \leftarrow
 \end{aligned}$$



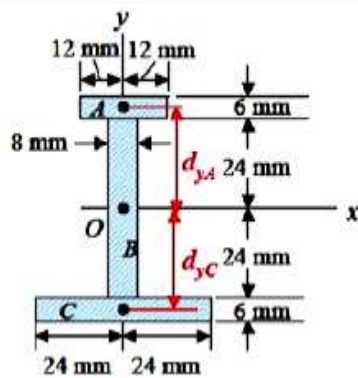
$$\begin{aligned}
 I_y &= (\bar{I}_y + Ad_x^2)_A + (\bar{I}_y + Ad_x^2)_B + (\bar{I}_y + Ad_x^2)_C \\
 &= \left[\frac{1}{12} (300)(100)^3 + (100 \times 300)(250)^2 \right] + \left[\frac{1}{12} (100)(600)^3 + 0 \right] \\
 &\quad + \left[\frac{1}{12} (300)(100)^3 + (100 \times 300)(250)^2 \right] \\
 &= 5.6 \times 10^9 \text{ mm}^4 \quad \leftarrow
 \end{aligned}$$

Example 6.6.

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes



Solution



$$\begin{aligned}
 I_x &= (\bar{I}_x + Ad_y^2)_A + (\bar{I}_x + Ad_y^2)_B + (\bar{I}_x + Ad_y^2)_C \\
 &= \left[\frac{1}{12} (24)(6)^3 + (24 \times 6)(27)^2 \right]_A \\
 &\quad + \left[\frac{1}{12} (8)(48)^3 + 0 \right]_B \\
 &\quad + \left[\frac{1}{12} (48)(6)^3 + (48 \times 6)(27)^2 \right]_C
 \end{aligned}$$

$$I_x = 390 \times 10^3 \text{ mm}^4 \quad \leftarrow$$

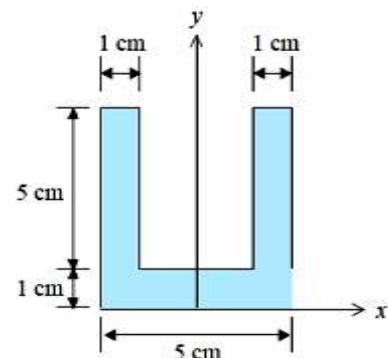
$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390 \times 10^3}{[(24 \times 6) + (8 \times 48) + (48 \times 6)]}} = 21.9 \text{ mm} \quad \leftarrow$$

$$\begin{aligned}
 I_y &= (\bar{I}_y + Ad_x^2)_A + (\bar{I}_y + Ad_x^2)_B + (\bar{I}_y + Ad_x^2)_C \\
 &= \left[\frac{1}{12} (6)(24)^3 \right]_A + \left[\frac{1}{12} (48)(8)^3 \right]_B + \left[\frac{1}{12} (6)(48)^3 \right]_C
 \end{aligned}$$

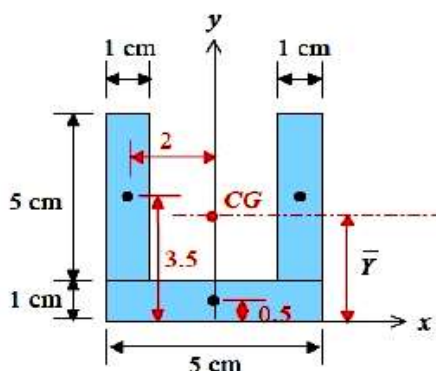
$$I_y = 64.3 \times 10^3 \text{ mm}^4 \quad \leftarrow \quad k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3 \times 10^3}{[(24 \times 6) + (8 \times 48) + (48 \times 6)]}} = 8.87 \text{ mm} \quad \leftarrow$$

Example 6.7.

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axis



Solution



$$\bar{Y} \sum A = \sum \bar{y} A$$

$$\begin{aligned}
 \bar{Y} &= \frac{2[(3.5)(5 \times 1)] + (0.5)(1 \times 5)}{3(5 \times 1)} \\
 &= 2.5 \text{ cm}
 \end{aligned}$$

• Moments of inertia about x axis

$$\begin{aligned}
 I_x &= 2\left[\left(\frac{1}{12}\right)(1)(5)^3 + (5 \times 1)(3.5)^2\right] + \frac{1}{3}(5)(1)^3 \\
 &= 145 \text{ cm}^4 \quad \leftarrow
 \end{aligned}$$

• Moments of inertia about centroid

$$\begin{aligned}
 \bar{I}_x &= I_x - Ad_y^2 \\
 &= 145 - (15)(2.5)^2 \\
 &= 51.25 \text{ cm}^4 \quad \leftarrow
 \end{aligned}$$

OR

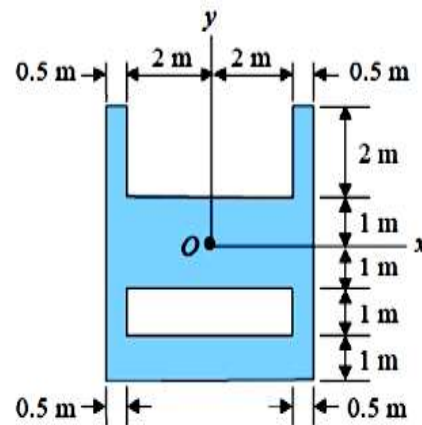
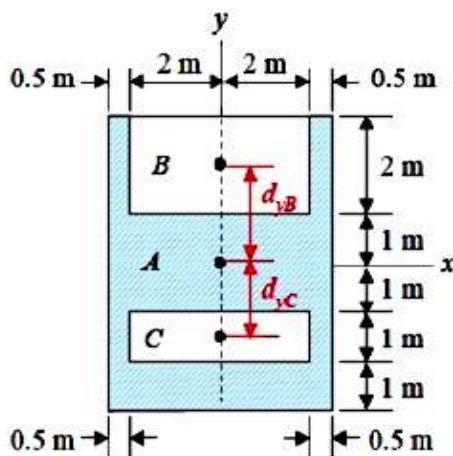
$$\begin{aligned}
 \bar{I}_x &= 2\left[\left(\frac{1}{12}\right)(1)(5)^3 + (5 \times 1)(1)^2\right] \\
 &\quad + \left[\left(\frac{1}{12}\right)(5)(1)^3 + (5 \times 1)(2)^2\right] \\
 &= 51.25 \text{ cm}^4 \quad \leftarrow
 \end{aligned}$$

$$\begin{aligned}
 \bar{I}_y &= I_y - 2\left[\left(\frac{1}{12}\right)(5)(1)^3 + (5 \times 1)(2)^2\right] + \frac{1}{12}(1)(5)^3 \\
 &= 51.25 \text{ cm}^4 \quad \leftarrow
 \end{aligned}$$

$$\bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848 \text{ cm} \quad \leftarrow$$

Example 6.8.

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes


Solution


$$I_x = (\bar{I}_x + A d_y^2)_A - (\bar{I}_x + A d_y^2)_B - (\bar{I}_x + A d_y^2)_C$$

$$= \left[\frac{1}{12} (5)(6)^3 + 0 \right]_A - \left[\frac{1}{12} (4)(2)^3 + (2 \times 4)(2)^2 \right]_B$$

$$- \left[\frac{1}{12} (4)(1)^3 + (4 \times 1)(1.5)^2 \right]_C$$

$$I_x = 46 \text{ m}^4 \quad \leftarrow$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{[(5 \times 6) - (4 \times 2) - (4 \times 1)]}} = 1.599 \text{ m} \quad \leftarrow$$

$$I_y = (\bar{I}_y + A d_x^2)_A - (\bar{I}_y + A d_x^2)_B - (\bar{I}_y + A d_x^2)_C$$

$$= \left[\frac{1}{12} (6)(5)^3 \right]_A - \left[\frac{1}{12} (2)(4)^3 \right]_B - \left[\frac{1}{12} (1)(4)^3 \right]_C$$

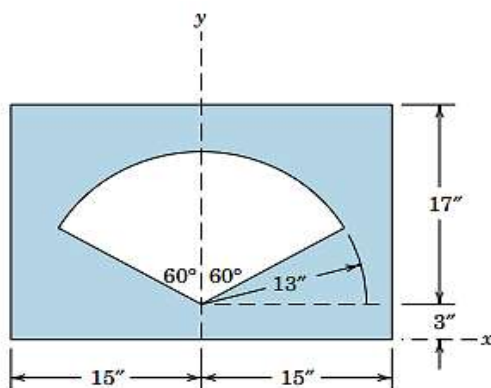
$$I_y = 46.5 \text{ m}^4 \quad \leftarrow$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{[(5 \times 6) - (4 \times 2) - (4 \times 1)]}} = 1.607 \text{ m} \quad \leftarrow$$

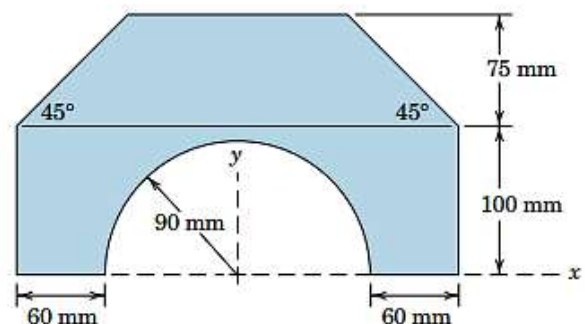
Exercise 6.2.

Determine the moment of inertia of the shaded area shown with respect to each of the coordinate axes

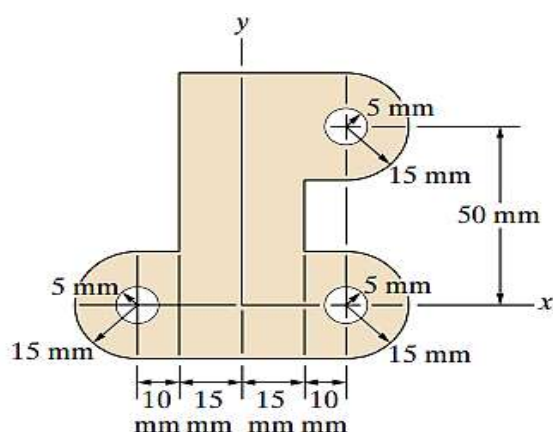
A.



C.



B.



Exercise 6.3.

Determine the product of inertia I_{xy} of the areas on Exercise 6.1 B and Exercise 6.2 C.

References

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